

The S Parameter in QCD from Domain Wall Fermions

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Abstract

We have computed the $SU(2)$ Low Energy Constant l_5 and the mass splitting between charged and neutral pions from a lattice QCD simulation of $n_f = 2 + 1$ flavors of Domain Wall Fermions at a scale of $a^{-1} = 2.33$ GeV. Relating l_5 to the S parameter in QCD we obtain a value of $S(m_H = 120 \text{ GeV}) = 0.42(7)$, in agreement with previous determinations. Our result can be compared with the value of S from electroweak precision data which constrains strongly interacting models of new physics like Technicolor. This work in QCD serves as a test for the methods to compute the S parameter with Domain Wall Fermions in theories beyond the Standard Model. We also infer a value for the pion mass splitting in agreement with experiment.

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I. INTRODUCTION

Models of dynamical electroweak symmetry breaking (DEWSB) like Technicolor are possible extensions of the Standard Model, and many proposals have been put forward for a strongly-interacting sector beyond the Standard Model since the original proposals in Refs. [1, 2]. Viable models of DEWSB must satisfy the constraints that follow from electroweak precision data [3, 4]. These constraints put severe limitations on Technicolor candidates, and QCD-like theories naively rescaled to the electroweak scale are already ruled out, see e.g. Refs. [5, 6] for recent reviews. The constraints are nicely encoded in bounds for the value of the S parameter introduced in Refs. [3, 7].

Theories near an infrared fixed point (IRFP) have been advocated as promising candidates for DEWSB. Preliminary attempts at estimating analytically the S parameter for these theories suggest that it is much reduced compared to QCD-like theories [8, 9]. Unfortunately these computations are not based on first principles, and have to rely on assumptions that are difficult to control. Early studies focused on finding evidence for an IRFP in theories with a large number of flavors in the fundamental representation of the gauge group following the seminal example in Ref. [10]. More recently, novel models have been proposed based on smaller numbers of fermions in higher-dimensional representations [11]. Several phenomenological scenarios have been proposed which build upon these ideas [12, 13].

Lattice simulations are now in a position to address these questions from first principles, so that the difficulties in dealing with the non-perturbative dynamics can be dealt with in a systematic way. The existence of IRFPs has been investigated in theories with fundamental fermions [14, 15, 16, 17, 18, 19, 20] and higher representations [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. These preliminary studies have mapped out the space of bare lattice parameters and are starting to study the spectrum of the candidate theories, and the flow of renormalized couplings. First results hint towards an interesting landscape of theories that could exhibit scale invariance at large distances.

Computing the S parameter in these theories from first principles is an important ingredient in trying to build successful phenomenological models. The S parameter is obtained on the lattice from the form factors that appear in the momentum-space VV-AA correlator, as defined below in Section II. Chiral symmetry plays an important role in guaranteeing the cancellation of power-divergent singularities when computing the above correlator. Hence lattice formulations that preserve chiral symmetry at finite lattice spacing are particularly well-suited for these studies. QCD is the ideal testing ground to develop and test the necessary lattice technology. In this work, we compute the S parameter in QCD with $n_f = 2 + 1$ flavors of Domain Wall fermions (DWF).

Our study closely follows the procedure described in Ref. [32] where the S parameter (or equivalently the $SU(3)$ Low Energy Constant L_{10}) was first computed from vacuum polarisation functions (VPFs) using overlap fermions. We adopt the method, apply it to Domain Wall Fermions, and widen its scope by using conserved currents and larger physical volumes.

Section II contains a short account of computational methods for the VPF on the lattice. Section III addresses the topics of power divergences and residual chiral symmetry breaking. In Section IV we present the numerical results obtained on the gauge configurations produced by the RBC and UKQCD collaborations using Domain Wall Fermions. Section V contains our results for the pion mass splitting. A discussion of the numerical results and a short conclusion can be found in Section VI.

II. VACUUM POLARISATION FUNCTIONS

Domain Wall Fermions are a five dimensional formulation of lattice QCD with an approximate chiral symmetry [33, 34, 35]. The residual explicit breaking of chiral symmetry appears in the Ward identities of the theory as terms proportional to the so-called residual mass m_{res} . The conserved vector and axial currents form a multiplet under this approximate lattice chiral symmetry.

The basic observables in this work are vacuum polarisation functions of the vector and the axial vector current. They are defined as current-current two-point functions in momentum space,

$$\Pi_{\mu\nu}^V(q) \equiv \sum_x e^{iq \cdot x} \langle 0 | \mathcal{V}_\mu(x) V_\nu(0) | 0 \rangle, \quad (1)$$

$$\Pi_{\mu\nu}^A(q) \equiv \sum_x e^{iq \cdot x} \langle 0 | \mathcal{A}_\mu(x) A_\nu(0) | 0 \rangle, \quad (2)$$

where \mathcal{V}_μ and \mathcal{A}_μ are the conserved vector and axial currents and V_μ and A_μ are the corresponding local currents. We consider local-conserved correlators from a new set of point source propagators with up to two units of spatial momentum. The definition of the conserved currents can be found in Ref. [35]. Since the Fourier transform in Eqs. (1,2) includes $x = 0$, power-divergent contributions can arise. Due to lattice chiral symmetry these divergences cancel in the difference of the vector and axial vector correlators, if conserved currents are used. Note that similar observables are used to compute hadronic contributions to the anomalous magnetic moment of the muon in Refs. [36, 37, 38], where one can find a more detailed discussion of the renormalization of the correlators. A similar cancellation was pointed out in Ref. [32], where overlap fermions have been used.

Following Ref. [32] we decompose the difference $\Pi_{\mu\nu}^{V-A} \equiv \Pi_{\mu\nu}^V - \Pi_{\mu\nu}^A$ into a longitudinal and a transverse part,

$$\Pi_{\mu\nu}^{V-A} = (q^2 \delta_{\mu\nu} - q_\mu q_\nu) \Pi^{(1)}(q^2) - q_\mu q_\nu \Pi^{(0)}(q^2). \quad (3)$$

For each momentum we average all components of $\Pi_{\mu\nu}$ which contribute to only one of the “polarisations”. For example for momenta with one non-zero direction q_κ we have $q^2 \Pi^{(0)} = \Pi_{\kappa\kappa}^{V-A}$ and $q^2 \Pi^{(1)} = \frac{1}{3} \sum_{\mu \neq \kappa} \Pi_{\mu\mu}^{V-A}$.

We use $SU(2)$ Chiral Perturbation Theory (ChPT) to fit to our data to extract the Low Energy Constant l_5^r . Since we are using $2+1$ flavor lattices the low-energy constants will implicitly depend on the strange quark mass. The more common notation in the literature is to use the $SU(3)$ LEC L_{10}^r which is related to l_5^r by

$$\begin{aligned} L_{10}^r &= l_5^r - \frac{1}{384\pi^2} \left(\log \frac{m_K^2}{\mu^2} + 1 \right), \\ &= l_5^r - 3 \cdot 10^{-5} \text{ for } \mu = m_\rho. \end{aligned} \quad (4)$$

The ChPT result for $\Pi^{(1)}$ can be found in Ref. [39]:

$$q^2 \Pi^{(1)}(m_\pi, q^2) = -f_\pi^2 - \left[\frac{1}{48\pi^2} \left(\bar{l}_5 - \frac{1}{3} \right) - \frac{1}{3} \sigma^2 \bar{J}(\sigma) \right] q^2 + O(q^4), \quad (5)$$

$$\text{with } \bar{J}(\sigma) = \frac{1}{16\pi^2} \left(\sigma \log \frac{\sigma - 1}{\sigma + 1} + 2 \right) \quad \text{and} \quad \sigma = \sqrt{1 - \frac{4m_\pi^2}{q^2}}. \quad (6)$$

With m_π , f_π known from fits to pseudoscalar and A_0 correlators (Table I) there is only one free parameter in (5) for each choice of the chiral scale μ . The scale invariant LEC \bar{l}_5 is defined by

$$\bar{l}_5 = -192\pi^2 l_5^r(\mu) - \log \frac{m_\pi^2}{\mu^2}, \quad (7)$$

and the corresponding convention for the S parameter [3] is

$$S = \frac{1}{12\pi} \left[-192\pi^2 l_5^r(\mu) + \log(\mu^2/m_H^2) - \frac{1}{6} \right]. \quad (8)$$

III. CONTACT TERMS IN WARD IDENTITIES

Contact terms in the Ward identities can yield finite contributions to the Fourier transform, so that $\Pi_{\mu\nu}^{V/A}$ is no longer transverse. However the contact terms cancel exactly in the difference between the vector–vector and the axial–axial correlator for DWF in the $L_s \rightarrow \infty$ and massless limit, provided the conserved/local correlators are used. Any power–divergent contribution also cancels in this difference as a result of the chiral symmetry of DWF.

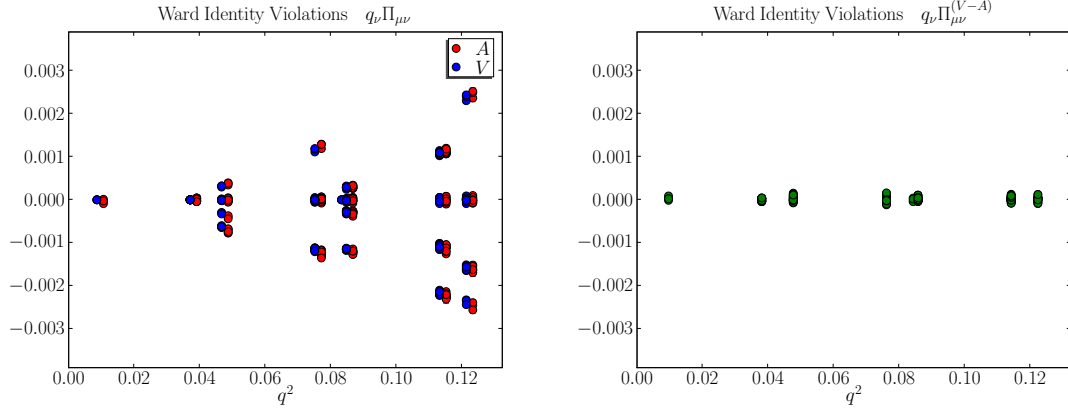


Figure 1: Ward identity violations in the chiral limit for the vector and axial vector currents (left) and their difference (right). The same scale is chosen in both plots to visualise the cancellation.

Following the notation introduced in the previous section we test the vector and axial Ward identities by extrapolating $q_\nu \Pi_{\mu\nu}^{V/A}$ to the chiral limit. We find that Ward identity violations are very similar for both currents and contact term contributions are greatly suppressed in the difference $\Pi_{\mu\nu}^{V-A}$, as shown in Figure 1. The cancellation in the chiral limit also hints at only small effects from the non-conservation of the axial current due to the residual mass of Domain Wall Fermions. The DWF axial current acquires a small multiplicative renormalisation which vanishes in the $L_s \rightarrow \infty$ limit [34]. Since our simulation is done at a fixed length in the fifth dimension ($L_s = 16$) we have to consider this effect as part of our error budget. In Refs. [40, 41, 42] it has been argued that $\Delta = |Z_A - 1|$ receives contributions from both delocalized modes above the mobility edge λ_c , and from localized near zero modes of H_W . The former is proportional to $e^{-\lambda_c L_s}$ while

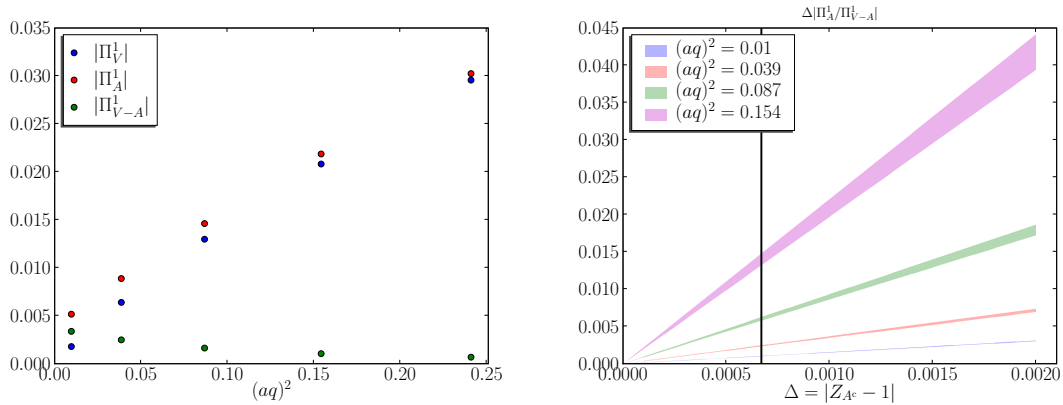


Figure 2: Left: cancellation between $\Pi^{(V)}$ and $\Pi^{(A)}$ as a function of q^2 , Right: relative error for given Δ for the lowest momenta.

the latter is proportional to $\frac{1}{L_s^2}$. These correspond to two different contributions to the residual mass: the exponential piece is linear in the corresponding component of m_{res} while the latter is quadratic in the localised contribution to m_{res} that is larger in our case.

As a pragmatic approach we vary Δ between 0 and $3am_{\text{res}}$ and estimate the relative error on the difference Π^1 as $\Delta \frac{\Pi^{(1),A}}{\Pi^{(1)}}$. Our conclusion is that there is no large cancellation for the local-conserved correlators since $q^2\Pi^{(1),V}$ approaches zero, as shown in Figure 2. We assume a conservative three percent systematic error in $\Pi^{(1)}$ for the non-conservation of the axial current.

IV. RESULTS

The data presented is from the ensembles generated by the RBC and UKQCD collaboration with the Iwasaki gauge action at $\beta = 2.25$ which corresponds to a lattice spacing of $a^{-1} = 2.33(4)$ GeV. The details of the ensembles will be published in [43]. We simulate with three values of the light quark mass which correspond to pion masses on the range of $290 \text{ MeV} \leq m_\pi \leq 400 \text{ MeV}$. Our chiral fits for l_5 rely on previous measurements of the pseudoscalar mass and decay constant at the unitary quark masses. For convenience we summarize the results obtained from correlators with gauge-fixed wall sources in Table I along with the vector and axial vector ground state masses which are used as a consistency check using Weinberg's sum rules and our data for l_5 . In our analysis

am_l	am_π	af_π	am_V	am_A
0.004	0.1269(4)	0.0619(3)	0.356(6)	0.522(13)
0.006	0.1512(3)	0.0645(3)	0.366(5)	0.543(18)
0.008	0.1727(4)	0.0671(3)	0.388(6)	0.551(9)

Table I: Meson masses and pseudoscalar decay constant in lattice units.

of $\Pi^{(1)}$ we include spatial momenta up to $(1, 1, 0)$ (or equivalent). We find excellent agreement between results with and without spatial momentum for the local-conserved $\Pi^{(1)}$, Figure 3. This is in contrast to the local-local data where there are two distinct branches of points.

As a first step we obtain effective values for L_{10} for each mass and momentum from our data for $\Pi^{(1)}$, Eq.(3). Our results are summarised in Table II. Since ChPT is known to have only a small radius of convergence in p^2 , we restrict ourselves to only the lowest momenta.

Now we perform a one parameter fit to obtain our central value for L_{10} . In Figure 4 the dependence of L_{10} on the momentum fit-range is shown for a fit including all three masses. In

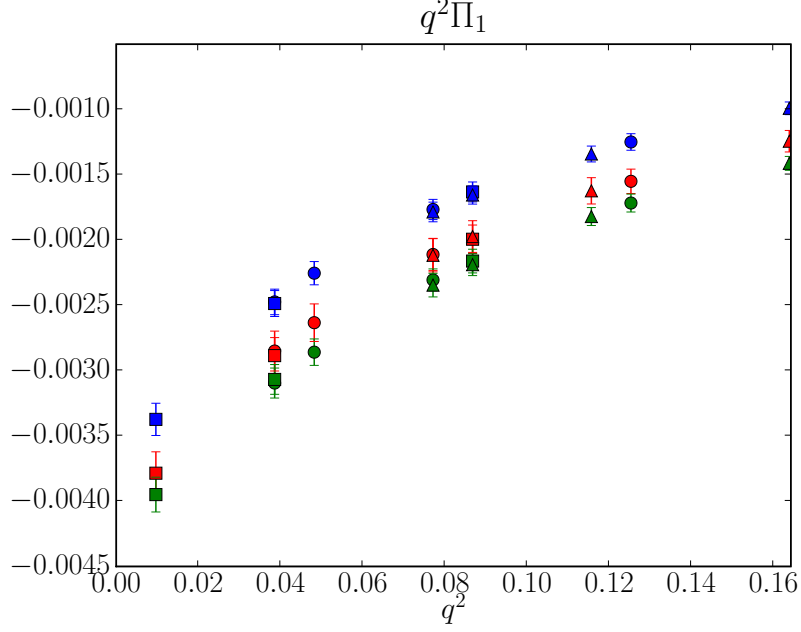


Figure 3: $q^2 \Pi^{(1)}$ at low momentum. The different symbols correspond to momenta of type $(n+1, 0, 0, 0)$: squares, $(n, 1, 0, 0)$: circles, $(n, 1, 1, 0)$: triangles with $n = 0, 1, 2$

the lower panel we observe that the $\chi^2/\text{d.o.f.}$ becomes larger than one when more than the lowest two momenta are included. Fits with fewer mass values exhibit the same behaviour. We however conclude that including only the lowest momentum gives a more reliable result since the higher momenta yield values for $q^2 \Pi^{(1)}$ which are consistently below the fit curve when including only the smallest momentum, Figure 5. Our central value therefore is obtained from a fit to all three masses and the lowest momentum only. In Figure 5 the data points included in the determination of L_{10} and the error band of the resulting fit is shown plotted against mass (left) and momentum (right). Our value for L_{10}^r is

$$L_{10}^r(\mu = 0.77 \text{ GeV}) = -0.0057(11)_{\text{stat}}(7)_{\text{sys}}. \quad (9)$$

The systematic error has several contributions: fit-range, lattice artefacts, scale setting, strange quark mass and finite volume, which are described in more detail below.

We estimate the error on the chiral fit by varying the fit-range in mass and q^2 and obtain an error of 0.0006, or 11%. The leading lattice artefacts with Domain Wall Fermions are parametrically of order $a^2 \Lambda_{\text{QCD}}^2$. We assume $\Lambda_{\text{QCD}} = 300 \text{ MeV}$, use $a^{-1} = 2.33(4) \text{ GeV}$ and double the result to obtain a three percent error. The uncertainty in the lattice scale is relevant for the definition of the chiral scale $\mu = m_\rho$. We vary a^{-1} within its error and obtain an additional two percent error. The dynamical strange quark mass in this computation is fixed to $am = 0.03$ which differs substantially

momentum	$(ap)^2$	am	$q^2\Pi^{(1)} \cdot 10^3$	$L_{10} \cdot 10^3$
(1, 0, 0, 0)	0.010	0.004	-3.37(12)	-5.6(1.6)
		0.006	-3.79(16)	-4.5(2.1)
		0.008	-3.95(13)	-7.0(1.7)
(2, 0, 0, 0)	0.039	0.004	-2.49(10)	-4.02(32)
		0.006	-2.88(14)	-3.89(45)
		0.008	-3.07(11)	-4.55(37)
(0, 1, 0, 0)	0.039	0.004	-2.5(10)	-3.99(32)
		0.006	-2.87(15)	-3.94(50)
		0.008	-3.22(10)	-4.06(34)
(1, 1, 0, 0)	0.048	0.004	-2.25(9)	-3.77(23)
		0.006	-2.63(14)	-3.74(37)
		0.008	-2.86(10)	-4.17(26)
(0, 1, 1, 0)	0.077	0.004	-1.77(8)	-3.07(13)
		0.006	-2.11(12)	-3.16(20)
		0.008	-2.31(8)	-3.51(14)

Table II: Results for $q^2\Pi^{(1)}$ and the resulting effective L_{10} for the lowest momenta and all three masses.

from the physical value. Using a reweighting procedure for the strange quark determinant [44] we vary the strange quark mass down to the physical point in lattice units and find variations in L_{10}^r of less than three percent. Possible finite volume effects (FVE) are well under control, since at the lightest mass we have $m_\pi L = 4$ and the estimate for FVE for f_π from [45] is 0.5%. Adding the errors in quadrature we find a total systematic error of 0.0007 which is dominated by the error on the chiral fit.

Comparing our result with the previous lattice determination [32] and phenomenological estimates [46, 47] we find it to be well consistent, Figure 6.

Lattice data for the lowest states in the vector and axial vector channels allows us to crosscheck

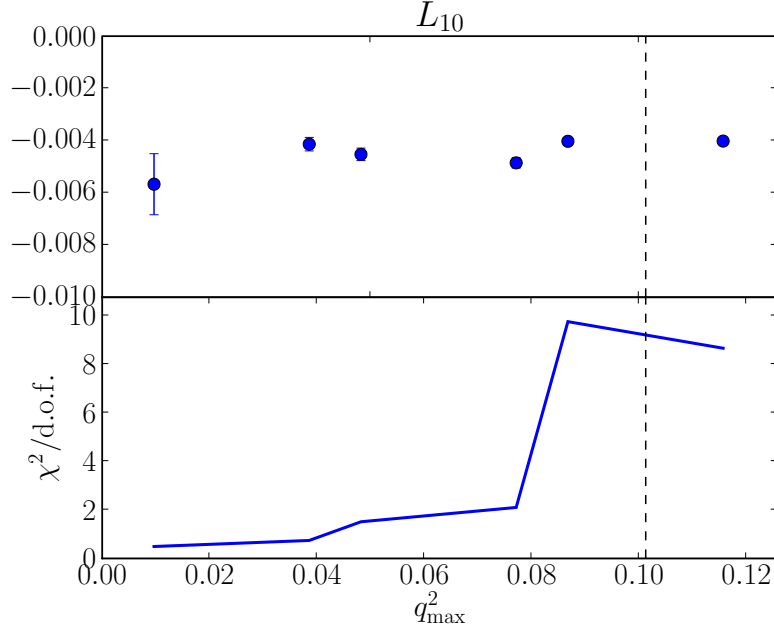


Figure 4: Upper panel: dependence of L_{10} on the fit-range in $(ap)^2$. Lower panel: reduced χ^2 of the corresponding fits described in the text. The dashed vertical line denotes the chiral scale in lattice units.

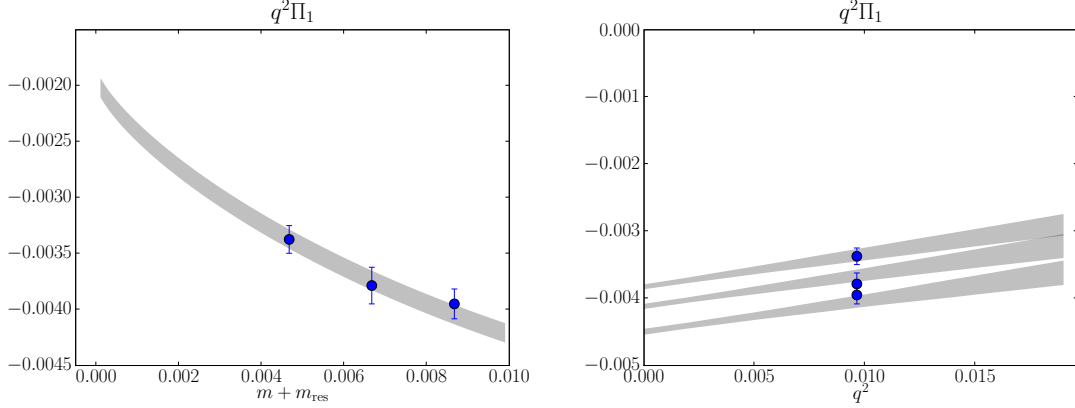


Figure 5: Data set and fit band used for the final result for L_{10} .

our result with the following sum rules for the spectral densities $\rho_{V/A}$ [48, 49],

$$\int ds s (\rho_V(s) - \rho_A(s)) = 0, \quad (10a)$$

$$\int ds (\rho_V(s) - \rho_A(s)) = f_\pi^2, \quad (10b)$$

$$\int ds \frac{1}{s} (\rho_V(s) - \rho_A(s)) = -8\bar{L}_{10}. \quad (10c)$$

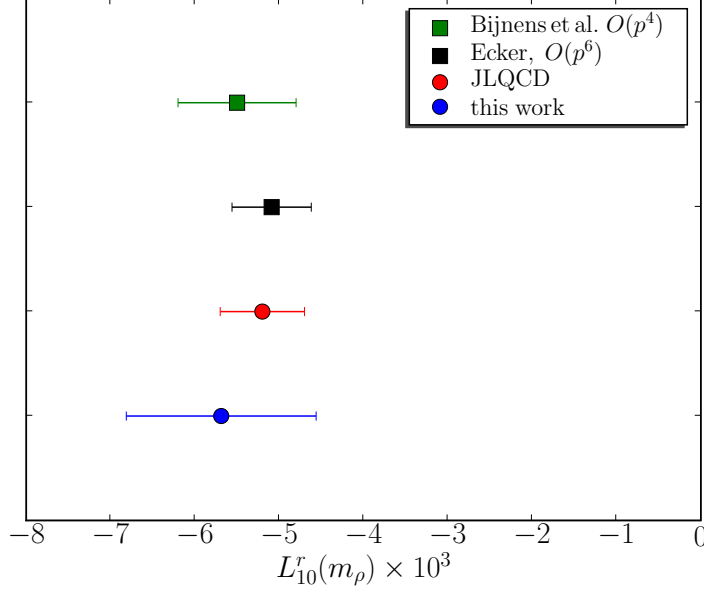


Figure 6: Comparison of L_{10}^r with other determinations.

We assume that the spectral densities are saturated by the lightest resonances,

$$\rho_{V/A} \approx f_{V/A}^2 \delta(s - m_{V/A}^2). \quad (11)$$

The resulting simplified sum rules are solved for \bar{L}_{10} using our values for m_V, m_A and f_π from Table I,

$$(f_V m_V)^2 - (f_A m_A)^2 = 0, \quad (12a)$$

$$f_V^2 - f_A^2 = f_\pi^2, \quad (12b)$$

$$\frac{f_V^2}{m_V^2} - \frac{f_A^2}{m_A^2} = -8\bar{L}_{10}. \quad (12c)$$

Thus we obtain an independent estimate for \bar{L}_{10} for each quark mass which we convert to $L_{10}^r(m_\rho)$ using the pions masses from Table I. In Figure 7, these results are shown by the black circles and are compared with our best estimate (9), indicated in the figure by the blue band. We find them to be well consistent. This agreement is a non-trivial finding since it is found at finite lattice spacing and without a chiral extrapolation for the values obtained from the sum rules (12a-12c).

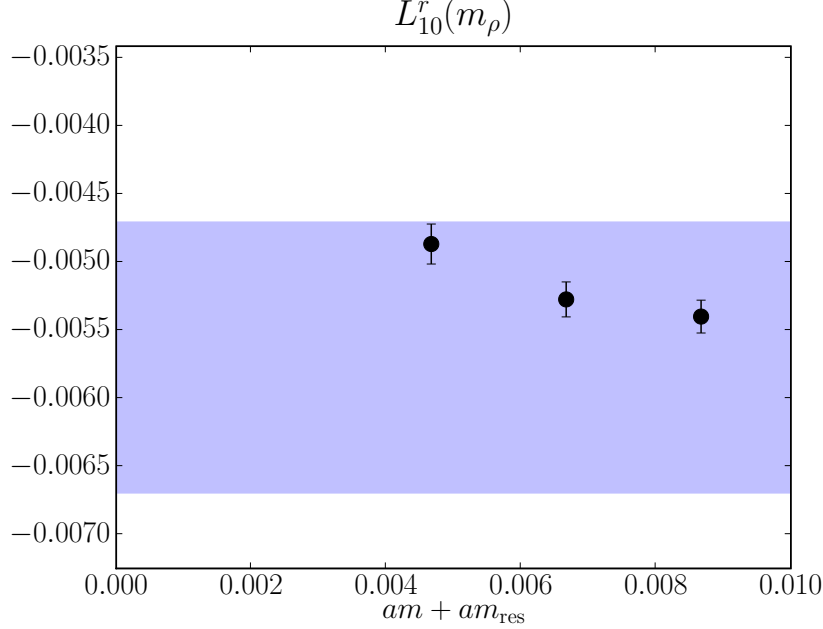


Figure 7: Results for L_{10}^r from the sum rules in Eq. (12) These are compared with the result from Eq. (9) which is indicated by the blue band.

V. PION MASS SPLITTING

The computation of the vacuum polarisation functions over the complete q^2 range allows the computation of the electromagnetic contribution to mass splitting between the charged and neutral pions [50],

$$m_{\pi^\pm}^2 - m_{\pi^0}^2 = -\frac{3\alpha}{4\pi} \int_0^\infty dq^2 \frac{q^2 \Pi^{(1)}}{f_\pi^2} = 1261 \text{ MeV}^2. \quad (13)$$

The simplest possible ansatz to describe the mass and momentum dependence of $\Pi^{(1)}$ which respects the sum rules (12a,12b) is

$$\begin{aligned} q^2 \Pi^{(1)}(q^2, m) &= -f_\pi^2 + \frac{f_V^2 q^2}{m_V^2 + q^2} - \frac{f_A^2 q^2}{m_A^2 + q^2}, \\ f_V &= x_1 + x_2 m_\pi^2, \quad m_V = x_3 + x_4 m_\pi^2, \\ f_A^2 &= f_V^2 - f_\pi^2, \quad m_A = m_V \frac{f_V}{f_A}, \end{aligned} \quad (14)$$

where x_1, \dots, x_4 are fit parameters. We include all data points up to $(aq)^2 < 1.0$ in the simultaneous fit to all three masses and obtain a stable fit with $\chi^2/\text{d.o.f.} = 1.04$. Extrapolating to the chiral limit we find

$$-\frac{3\alpha}{4\pi} \int_0^1 dq^2 \frac{q^2 \Pi^{(1)}}{f^2} = 1040(220) \text{ MeV}^2. \quad (15)$$

We have also tested the two different fit forms used in [32] which include an additional term, which is relevant at high q^2 , however we do not find a significant change in the fit parameters x_1, \dots, x_4 or a smaller $\chi^2/\text{d.o.f.}$ when we include these additional fit parameters. An illustration of

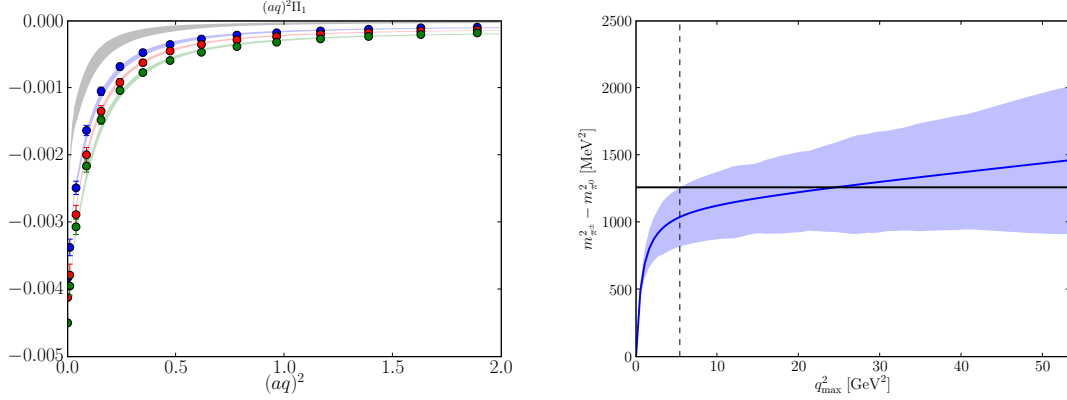


Figure 8: Left: fit with ansatz (14), where the grey band denotes the result in the chiral limit. Right: mass splitting $m_{\pi^\pm}^2 - m_{\pi^0}^2$ as a function of the cut-off momentum q_{max}^2 , the horizontal line denotes the physical value, the dashed line marks our chosen cut-off.

the dependence on the cut-off is shown in Figure 8. The remaining part of the integral is estimated under the assumption that the functional dependence follows the asymptotic behaviour for large q^2 ,

$$\Pi^{(1)} \rightarrow \frac{c}{q^6}. \quad (16)$$

Matching (16) to our fit result in the chiral limit we obtain $c = -0.0062(58) \text{ GeV}^6$. Here we have doubled the error to allow for possible deviations from the asymptotic form. The remaining integral is then

$$-\frac{3\alpha}{4\pi} \int_1^\infty dq^2 \frac{q^2}{\Pi^{(1)}} f^2 = 140(130) \text{ MeV}^2, \quad (17)$$

yielding a pion mass splitting of

$$m_{\pi^\pm}^2 - m_{\pi^0}^2 = 1180(260) \text{ MeV}^2. \quad (18)$$

This error does not include a rigorous investigation of the systematic uncertainties as was done in Sec. IV for L_{10} , however we expect these uncertainties to be well within the large statistical error of twenty percent obtained from the integral (15) with $0 < (aq)^2 \leq 1$. The value we obtain is in excellent agreement with the experimental value $m_{\pi^\pm}^2 - m_{\pi^0}^2 = 1261 \text{ MeV}^2$ [51].

Finally we can use the result of the fit defined in Eq. (14) to obtain another independent determination for L_{10} . From the slope of the ansatz (14) in the chiral limit, we find $L_{10}^r(m_\rho) = 5.19(14) \cdot 10^{-3}$ which is in agreement with our ChPT analysis. This is a nice check of the saturation of the Weinberg sum rules. Unfortunately the large error on our determination of l_5 does not allow a more quantitative comparison.

VI. CONCLUSIONS

We have computed the S parameter in QCD using the gauge configurations generated by the RBC-UKQCD collaboration for 2+1 flavors of dynamical DWF fermions. According to the procedure outlined in Ref. [32], the S parameter is extracted from the form factor that appears in the parametrization of the VV-AA correlator. Our final result is

$$L_{10}^r(\mu = 0.77 \text{ GeV}) = -0.0057(11)_{\text{stat}}(7)_{\text{sys}}; \quad (19)$$

where the error is still dominated by the statistical precision of our simulation. On the other hand, the systematic error is dominated by the choice of the fit-range used in the chiral extrapolation. A further improvement to these simulations would be to include partially twisted boundary conditions, enabling access to smaller momenta where ChPT can be employed reliably. Using results from meson spectroscopy, we were able to check the saturation of the Weinberg sum rule by the lowest-lying resonances in QCD. At the current level of accuracy, we find that the contribution from the lowest-lying resonances accounts for the total value of the S parameter, which confirms a widely-used assumption in phenomenological studies. Our best estimate for the S parameter (8) with a Higgs boson mass of $m_H = 120 \text{ GeV}$ is

$$S = 0.42(7), \quad (20)$$

where we have rescaled the renormalization scale μ with the ratio of the pion decay constant to the Higgs vacuum expectation value $v = 246 \text{ GeV}$.

From the same correlators we have extracted the electromagnetic pion mass splitting, and the result $\Delta m_\pi^2 = 1180 \pm 260 \text{ MeV}^2$ is in excellent agreement with the experimental results. Despite the large error, the present computation shows that it is possible to extract this quantity from the currently available DWF ensembles.

The results presented here are only made possible through the chiral properties of DWF which ensure the cancellation of the power-divergent contributions in the current correlators. The QCD

analysis we have performed lays the groundwork for the computation of the S parameter in potential Technicolor model candidates such as those proposed Refs. [13, 52]. Computing the S parameter in Technicolor theories is a crucial step in identifying the models that survive the constraints from electroweak precision measurements. We plan to extend this computation to the models recently simulated in Refs. [24].

Acknowledgments

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